Math 564: Advance Analysis 1
Lecture 17

Infinite products of measure spaces. We already discussed products of two measure spaces, hence we can define finite products of measure spaces by induction or just repeat the construction/arguments we did but for a finite product.
What about infinite products?

Let I be an index set (e.g. $I := \mathbb{N}$ but also $I := \mathbb{R}$). For each $i \in I$, let $(X_i, \mathcal{B}_i, \mu_i)$ be measure spaces. Firstly, let $X := \prod_{i \in I} X_i$, and let $\bigotimes_{i \in I} \mathcal{B}$ denote the σ-algebra generated by cylinders, i.e. sets of the form

$$[(B_i)_{i \in I_0}] := \prod_{i \in I_0} B_i \times \prod_{j \in I \setminus I_0} X_j,$$

where $I_0 \subseteq I$ is finite, $B_i \in \mathcal{B}_i \ \forall i \in I_0$.
We would like to get a measure μ on $\bigotimes_{i \in I} \mathcal{B}_i$ s.t.

$$\mu\left([(B_i)_{i \in I_0}]\right) = \prod_{i \in I_0} \mu_i(B_i) \times \prod_{j \in I \setminus I_0} \mu_j(X_j)$$

for each cylinder $[(B_i)_{i \in I_0}]$. Firstly, we want this infinite product to be well-defined. Also, we don't want it to be ∞ or 0 to get an interesting measure. In particular, we want all but finitely many measures $\mu_i(X_i)$ to be finite. To ensure this well-defined we only consider probability spaces.
So assuming $\mu(X_i) = 1 \ \forall i \in I$, we want a measure:

$$\mu\left([(B_i)_{i \in I_0}]\right) = \prod_{i \in I_0} \mu_i(B_i). \qquad (*)$$

Let $\mathcal{A}$ denote the algebra generated by cylinders, which is just the collection of finite disjoint unions of cylinders.
Use $(*)$ to define a finitely additive measure μ on $\mathcal{A}$.
Its well-definedness and hence also finite additivity is shown as usual via taking refinements of partitions.
So now μ is a finitely additive measure on $\mathcal{A}$, so it's ctbly sup additive, i.e. if $A = \bigsqcup_{n \in \mathbb{N}} A_n$ where $A, A_n \in \mathcal{A}$, then

$$\mu(A) \geq \sum_{n \in \mathbb{N}} \mu(A_n).$$

Theorem (Kakutani 1943). μ is also ctbly subadditive. In particular, by Carathéodory, μ admits an extension to a probability measure on $\bigotimes_{i \in I} \mathcal{B}_i$.

We will prove this in the following special case:

Prop. For $I := \mathbb{N}$ and each $(X_i, \mathcal{B}_i, \mu_i)$ being a standard prob. space, i.e. X_i is Polish and $\mathcal{B}_i = \mathcal{B}(X_i)$.

Proof. By the Borel Isomorphism theorem, we may assume X_i is a compact Polish space (e.g. $2^{\mathbb{N}}$, $[0,1]$, $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$).
To show that μ defined above on the algebra $\mathcal{A}$ generated by cylinders is subadditive, it's enough to let $A = \bigsqcup_{n \in \mathbb{N}} A_n$ a disjoint union of cylinders, where A itself is a cylinder $A := [(B_i)_{i<N}]$. Now we use the tightness of the μ_i to replace each B_i with a compact $B_i' \subseteq B_i$ so $A' := [(B_i')_{i<N}]$ is compact by Tychonoff's Theorem and $\mu(A) \approx_{\varepsilon/2} \mu(A')$. Similarly, using the regularity of each μ_i, we can replace each cylinder A_n with an open cylinder $\hat{A}_n \supseteq A_n$ with $\mu(\hat{A}_n) \approx_{\frac{\varepsilon}{4 \cdot 2^n}} \mu(A_n)$,

so $\{\overset{\circ}{\tilde{A}}_n\}_{n\in\mathbb{N}}$ is an open cover of the compact set A', hence admits a finite subcover $\{\hat{A}_0, \hat{A}_1, \dots, \hat{A}_k\}$, which shows that

$$\mu(A) \approx_{\varepsilon/2} \mu(A') \leq \sum_{i=0}^{k} \mu(\hat{A}_i) \approx_{\varepsilon/2} \sum_{i=0}^{k} \mu(A_i). \qquad \square$$

Differentiation of measures.

Given two measures μ and ν on the same measurable space $(X, \mathcal{B})$, we would like to understand how they relate to each other. We already defined the notion of absolute continuity $\mu \ll \nu$, i.e. for all $B \in \mathcal{B}$, $\nu(B) = 0 \Rightarrow \mu(B) = 0$. We also recall that when μ is finite then a Borel-Cantelli argument shows that $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $\nu(B) \leq \delta$ then $\mu(B) \leq \varepsilon$, which justifies the term absolute continuity.

The opposite notion to absolute continuity is orthogonality.

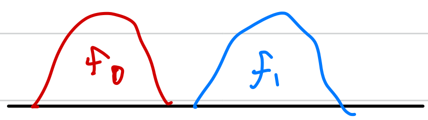
Def. We say that μ and ν are *orthogonal*, denoted $\mu \perp \nu$, if $X = X_\mu \sqcup X_\nu$ s.t. $\mu(X_\nu) = 0$ and $\nu(X_\mu) = 0$.
If this is the case, then it is easy to check that this decomposition is unique up sets that are both μ and ν null, i.e. if $X = X_\mu' \sqcup X_\nu'$ is another such decomp. then $X_\mu \triangle X_\mu'$ and $X_\nu \triangle X_\nu'$ are null wrt both measures.

Examples. (a) A pointmass δ_0 at 0 is orthogonal to the Lebesgue measure, with decomposition $\mathbb{R} = \{0\} \sqcup (\mathbb{R} \setminus \{0\})$.
Consequently, any ctbl positive linear combination of Dirac measures is orthogonal to the Lebesgue measure.

(b) Let μ_p be the Bernoulli (p) measure on $2^{\mathbb{N}}$. Let $h: 2^{\mathbb{N}} \hookrightarrow C$ be the canonical homeomorphism from $2^{\mathbb{N}}$ to the standard $\frac{1}{3}$ Cantor set, i.e. $h(x) := \sum_{n \in \mathbb{N}} 3^{-n} \cdot 2 \cdot x(n)$. Let $\nu_p := h_* \mu_p$.

$\lambda(C) = 0$ but $\nu_p(C) = 1$ while $\nu_p(\mathbb{R} \setminus C) = 0$. So $\nu_p \perp \lambda$ and the decomposition is $C \sqcup (\mathbb{R} \setminus C)$.

(c) For a fixed measure μ, let $X = X_0 \sqcup X_1$ and let $f_i : X \to [0, \infty]$ be any measurable functions $f_i = f_i \cdot \mathbb{1}_{X_i}$.
Then the measures μ_{f_0} and μ_{f_1} are orthogonal with $X = X_0 \sqcup X_1$ being the decomposition.



Lebesgue decomposition theorem. For any two σ-finite measures μ, ν on a measurable space $(X, \mathcal{B})$, $X := X_1 \sqcup X_0$ s.t. $\mu|_{X_1} \ll \nu|_{X_1}$ and $\mu|_{X_0} \perp \nu|_{X_0}$.

Def. Measures μ and ν on the same measurable space $(X, \mathcal{B})$ are called equivalent, denoted $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$. This means that μ and ν have the same null sets.

From Lebesgue decomp. thm, we get:

Cor. For any two σ-finite measures μ and ν, $\exists\ X = X_0 \sqcup X_1$ such that $\mu|_{X_0} \perp \nu|_{X_0}$ and $\mu|_{X_1} \sim \nu|_{X_1}$.

Proof. HW.

To present the proof of Lebesgue decomposition nicely, also for other purposes later, we would like to be able to subtract one measure from the other, which yields signed measures.

Def. A function $\mu : \mathcal{B} \to [-\infty, \infty]$ on a σ-algebra $\mathcal{B}$ on X is called a signed measure if

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\bigsqcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n)$

(iii) μ doesn't attain both $-\infty$ and ∞ values.

Examples. (a) All measures are signed measures.

(b) If μ, ν are measures where at least one is finite, then $\mu - \nu$ is a signed measure.

(c) Let $f \in L^1(X, \mu)$, then $\mu_f(B) := \int f \, d\mu$ is a finite signed measure. Note that $\mu_f = \mu_{f^+} - \mu_{f^-}$, like in (b).

The following shows that (b) contains all examples of signed measures.

Jordan Decomposition Theorem. Every signed measure is of the form $\mu - \nu$, where μ, ν are measures one of which is finite.